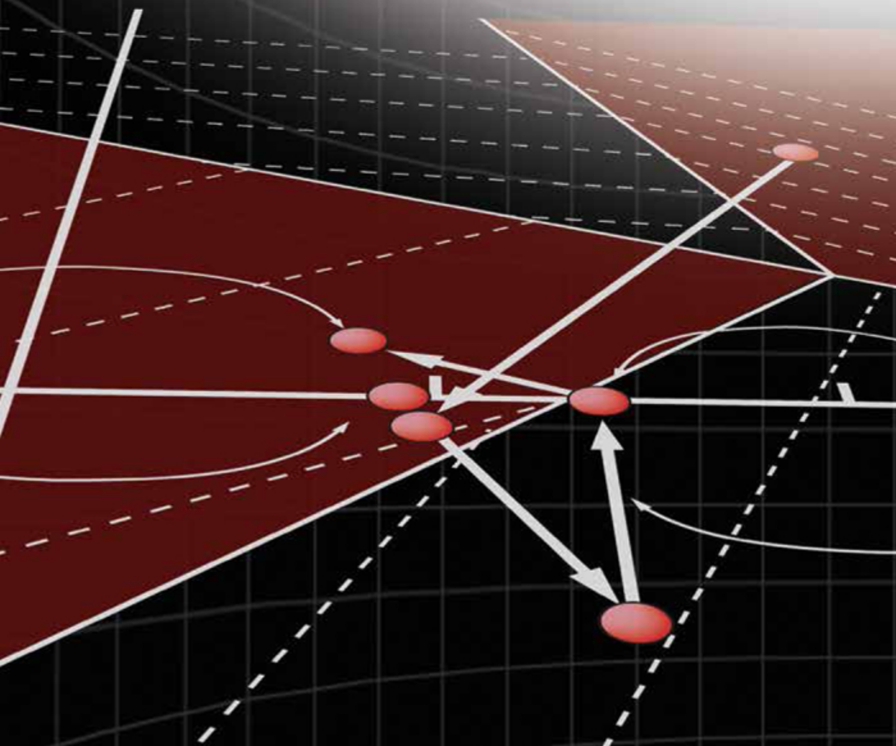
Ronald L. Rardin

# Optimization In Operations Research

Second Edition



**Primers**



**Algorithms**

# Optimization in Operations Research

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**SECOND EDITION**

**RONALD L. RARDIN**

*University of Arkansas*

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# Preface

It is now nearly two decades since publication of the first edition of my textbook ***Optimization in Operations Research***. Since that time thousands of students and hundreds of instructors, researchers, and practitioners have had the opportunity to benefit from its consistent content and accessible design. Of course, not all have seen benefit, but many have written kind reviews and letters expressing their high regard for the book. Also, the Institute of Industrial Engineers honored it with their Joint Publishers Book-of-the-Year Award in 1999.

In this second edition, I have tried to preserve what was best about the original while updating it with new and enhanced content. The goal remains the same—to make the tools of optimization modeling and analysis accessible to advanced undergraduate and beginning graduate students who follow the book in their studies, as well as researchers and working practitioners who use it as a reference for self-study. Emphasis is on the skills and intuitions they can carry away and apply in real settings or later coursework.

Although aimed at that same goal, much is new in the second edition:

- Stochastic optimization is covered for the first time with Stochastic Programming in Chapter 4, and Markov Decision Processes in Chapter 9.
- Coverage of linear programming techniques is expanded in Chapter 6 to encompass dual and primal-dual methods.
- New sections rigorously formalize optimality conditions for linear programming in Chapter 6, and cutting plane theory in Chapter 12.
- Treatment of the Hungarian Algorithm for assignment, and min/max spanning tree methods has been added to Chapter 10.
- A whole new Chapter 13 is devoted to large-scale optimization techniques including Delayed Column Generation, Lagrangian Relaxation, Dantzig–Wolfe Decomposition, and Benders' Partitioning.
- A whole new Chapter 14 treats the theory of computational complexity to provide a rigorous foundation for comparing problems and algorithms.
- Nonlinear Chapter 17 now includes coverage of the popular Sequential Quadratic Programming method.
- More generally, additional mathematical rigor is added to justifications of methods throughout the book, including tracking computational orders for most.

New topics seek to cover even more completely the full breadth of optimization (or mathematical programming) that might be of interest to the book's intended audience. Those span linear, integer, nonlinear, network, and dynamic programming models and algorithms, in both single and multi-objective context, and a rich sample of domains where they have been applied.

With content so inclusive, it is important to recognize that almost no reader or course will ever use it all, much less in the exact sequence presented in the book. For that reason, I have tried to make the organization of material as transparent and re-entrant as possible.

Dependencies between sections are minimized and clearly identified with explicit references. One- and two-page **Primers** concisely review prerequisite material where it is needed in the development to save diversions to other sources. To keep the focus on intuitions and strategies behind topics, **Definitions**, **Principles** and **Algorithms** are set out in easy-to-spot boxes, where high-level ideas can be located and absorbed quickly. When more detail is of interest, computations and discussions that may extend to several pages are recapped immediately in concise **Examples** (also marked for easy identification). For readers and instructors seeking more reinforcement with **Exercises** at the end of chapters, convenient icons clearly tag which of those require computer software () or advanced calculators (), and which have answers provided at the back of the book ().

The new edition also builds on my firm belief that making optimization materials accessible and exciting to readers of diverse backgrounds requires making the book a continuing discourse on optimization modeling. Every algorithm and analytic principle is developed in the context of a brief story set out as an **Application**. Also, computational exercises often begin with a formulation step. Many of those stories are derived from real OR applications footnoted in the development. Story settings—however contrived—provide a context for understanding both the needed decision variables, constraints and objectives of model forms, and steps in computation. For example, ideas like improving directions are more intuitive if some quantity in a story, not just a mathematical function, is clearly getting better as an algorithm is pursued. Likewise, binary decision variables become intuitive if the reader can see the either-or nature of some application decisions.

A related conviction is that students cannot really learn any mathematical topic without working with it in homework exercises. That is why the second edition continues the tradition of the first in providing a full range of exercises at the end of each chapter. Some continue from the first edition, but many are new or posed over modified parameter values. The range of exercises begins with verifications of algorithm details, definitions and properties, which are essential to building intuition about the methods. But a range of formulation exercises is also included extending from tiny examples subject to graphic or inspection solution to more complex applications drawn from real OR work that challenge formulation skills. In addition, a new Group Projects appendix details assignments I have used for years to engage student teams more deeply in published reports of actual optimization applications.

Early introductory books in optimization focused heavily on hand application of algorithms to compute solutions of tiny examples. With almost all real optimization now done with the help of large-scale computer software, more recent sources have sometimes limited attention to formulating data sets for submission to one of those algorithms—treating the computation largely as a black box.

I reject both these extremes. Graphic solution of small instances and hand implementation of algorithmic methods are essential if students are to internalize the principles on which the computation is based. The second edition continues my earlier pattern of moving quickly to such intuitive examples as each new concept is introduced. At the same time, no reader will ever grow excited about the power of optimization methods if he or she sees them applied only to tiny examples, much less abstract mathematical forms. That is why many of the examples and exercises in both the first and second editions of the book ask students to apply available class software on models of greater size, where answers are not apparent until formal methods are shown to reveal them. Brief sections have also been added on coding models for software like AMPL.

Perhaps the greatest challenge in trying to bridge undergraduate and beginning graduate audiences in optimization is the question of mathematical rigor. Elementary treatments simply introduce algorithmic mechanics with little if any argument for their correctness. On the other hand, more advanced books on optimization methods often devolve quickly into rigorous mathematical propositions and formal proofs with almost no discussion of underlying strategies, intuitions, and tractability.

My effort in the first edition was to bridge that gap by focusing on the intuitions and strategies behind methods, and on their relative tractability, while offering only limited arguments for their correctness. In the interest of better serving the introductory graduate and self-study audiences, the second edition adds significantly more rigor to the arguments presented. They are still not stated in theorem or proof format, but most key elements of rationales are now justified.

I am proud of how the long overdue second edition has emerged, and I hope readers will agree that it is a significant advance over the first. I look forward to your comments as the new developments are absorbed.

I want to thank deeply the hundreds of students, friends, and colleagues at Georgia Tech, Purdue and the University of Arkansas for their advice and encouragement as the new edition has taken shape. This goes especially for a series of Graduate Assistants who have helped with exercises and solutions, and for the patience and support of department heads Marlin Thomas, Dennis Engi, John English, Kim Needy, and Ed Pohl. Finally, I need to thank my family—especially my wife Blanca and my son Rob—for their patience and encouragement in my long slog to finish the task.

# About the Author

**Dr. Ronald L. (Ron) Rardin** retired as Distinguished Professor Emeritus in 2013 after a 40-year record of leadership as an educator and researcher in optimization methods and their application culminating after 2007 as John and Mary Lib White Distinguished Professor of Industrial Engineering on the faculty of the University of Arkansas-Fayetteville. He headed the University's Center on Innovation in Healthcare Logistics (CIHL) targeting supply chain and material flow aspects of healthcare operations in collaboration with a variety of healthcare industry organizations. He also took the lead with colleagues at Arkansas in founding the Health Systems Engineering Alliance (HSEA) of industrial engineering academic programs interested in healthcare.

Earlier, Professor Rardin retired in 2006 as Professor Emeritus of Industrial Engineering at Purdue University after completing 24 years there, including directing the Purdue Energy Modeling Research Groups, and playing a leading role in Purdue's Regenstrief Center for Healthcare Engineering. Previously he had served on the Industrial and Systems Engineering faculty at the Georgia Institute of Technology for 9 years. He also served the profession in a rotation from 2000–2003 as Program Director for Operations Research and Service Enterprise Engineering at the National Science Foundation, including founding the latter program to foster research in service industries.

Dr. Rardin obtained his B.A. and M.P.A. degrees from the University of Kansas, and after working in city government, consulting and distribution for five years, a Ph.D. at Georgia Institute of Technology.

His teaching and research interests center on large-scale optimization modeling and algorithms, especially their application in healthcare and energy. He is an award winning teacher of those topics, and co-author of numerous research papers and two comprehensive textbooks: a graduate text *Discrete Optimization*, published in 1988, and a comprehensive undergraduate textbook on mathematical programming, *Optimization in Operations Research*, which was published in 1998 and received the Institute of Industrial Engineers (IIE) Book of the Year award. Among his many other honors, he is a Fellow of both IIE and the Institute for Operations Research and the Management Sciences (INFORMS), as well as 2012 winner of the IIE's David F. Baker award for career research achievement.

# Optimization in Operations Research

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# CHAPTER 1

# Problem Solving with Mathematical Models

Any student with the most elementary scientific training has encountered the idea of solving problems by analyzing mathematical equations that approximate the physical realities of the universe we inhabit. Countless questions about objects falling, beams shearing, gases diffusing, currents flowing, and so on, are reduced to simple computations upon skillful application of one of the natural laws passed to us by Newton, Ohm, Einstein, and others.

The applicable laws may be less enduring, but "operations" problems such as planning work shifts for large organizations, choosing investments for available funds, or designing facilities for customer service can also be posed in mathematical form. A **mathematical model** is the collection of variables and relationships needed to describe pertinent features of such a problem.

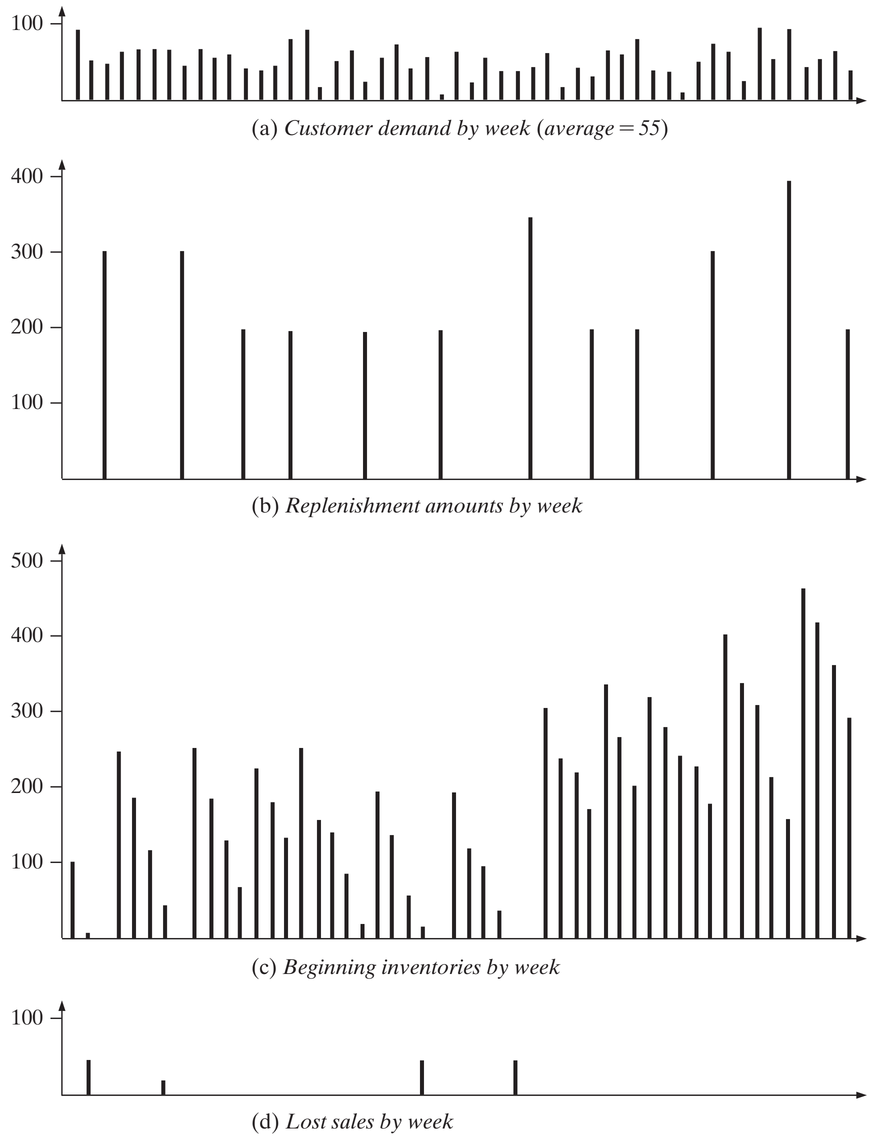
> Definition 1.1 **Operations research (OR)** is the study of how to form mathematical models of complex engineering and management problems and how to analyze them to gain insight about possible solutions.

In this chapter some of the fundamental issues and vocabulary related to operations research are introduced.

## 1.1 OR Application Stories

Operations research techniques have proved useful in an enormous variety of application settings. One of the goals of this book is to expose students to as broad a sample as possible. All application examples, many end-of-chapter exercises, several complete sections, and three full chapters present and analyze stories based on OR applications.

Whenever possible, these problems are drawn from reports of real operations research practice (identified in footnotes). Of course, they are necessarily reduced in size and complexity, and numerical details are almost always made up by the author.




(a) *Customer demand by week (average = 55)*

(b) *Replenishment amounts by week*

(c) *Beginning inventories by week*

(d) *Lost sales by week*

**FIGURE 1.1** Mortimer Middleman Example History

Other stories illustrate key elements of standard applications but greatly oversimplify, to facilitate quick learning.

A handful of continuing examples are even smaller and more contrived. They still have a story, but convenience in illustrating methodological issues takes precedence over reality of application.

## APPLICATION 1.1: MORTIMER MIDDLEMAN

Our first story is of the totally made-up variety. Mortimer Middleman—friends call him MM—operates a modest wholesale diamond business. Several times each year

MM travels to Antwerp, Belgium, to replenish his diamond supply on the international market. The wholesale price there averages approximately \$700 per carat, but Antwerp market rules require him to buy at least 100 carats each trip. Mortimer and his staff then resell the diamonds to jewelers at a profit of \$200 per carat. Each of the Antwerp trips requires 1 week, including the time for Mortimer to get ready, and costs approximately \$2000.

Customer demand values in Figure 1.1(a) show that business has been good. Over the past year, customers have come in to order an average of 55 carats per week.

Part (c) of Figure 1.1 illustrates Mortimer's problem. Weekly levels of on-hand diamond inventory have varied widely, depending on the ups and downs in sales and the pattern of MM's replenishment trips [Figure 1.1(b)].

Sometimes Mortimer believes that he is holding too much inventory. The hundreds of carats of diamonds on hand during some weeks add to his insurance costs and tie up capital that he could otherwise invest. MM has estimated that these holding costs total 0.5% of wholesale value per week (i.e., $0.005 \times \$700 = \$3.50$ per carat per week).

At other times, diamond sales—and Mortimer's \$200 per carat profit—have been lost because customer demand exceeded available stock [see Figure 1.1(d)]. When a customer calls, MM must either fill the order on the spot or lose the sale.

Adding this all up for the past year, MM estimates holding costs of \$38,409, unrealized profits from lost sales of \$31,600, and resupply travel costs of \$24,000, making the annual total \$94,009. Can he do better?

## 1.2 OPTIMIZATION AND THE OPERATIONS RESEARCH PROCESS

Operations research deals with **decision problems** like that of Mortimer Middleman by formulating and analyzing mathematical models—mathematical representations of pertinent problem features. Figure 1.2 illustrates this OR process.

The process begins with formulation or modeling. We define the variables and quantify the relationships needed to describe relevant system behavior.

Next comes analysis. We apply our mathematical skills and technology to see what conclusions the model suggests. Notice that these conclusions are drawn from

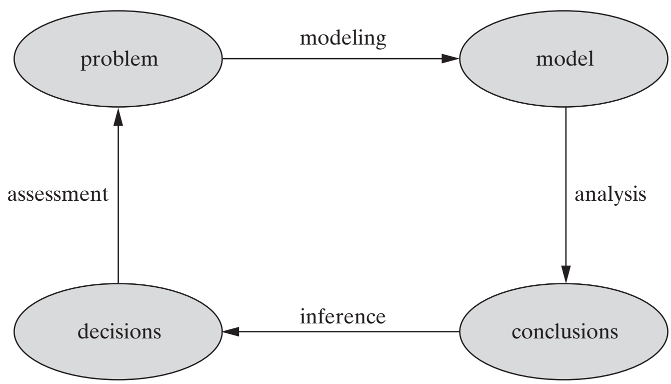



**FIGURE 1.2** Operations Research Process

the model, not from the problem that it is intended to represent. To complete the process, we must engage in inference, that is, argue that conclusions drawn from the model are meaningful enough to infer decisions for the person or persons with the problem.

Often, an assessment of decisions inferred in this way shows them to be too inadequate or extreme for implementation. Further thought leads to revised modeling, and the loop continues.

## *Decisions, Constraints, and Objectives*

We always begin modeling by focusing on three dimensions of the problem:

> Definition 1.2 The three fundamental concerns in forming operations research models are (a) the **decisions** open to decision makers, (b) the **constraints** limiting decision choices, and (c) the **objectives** making some decisions preferred to others.

In dealing with virtually any decision problem—engineering, management, or even personal—explicitly defining the decisions, constraints, and objectives helps to clarify the issues. Mortimer is obviously the decision maker in our diamond inventory management example. What decisions does he get to make?

Actually, MM makes hundreds of decisions each year about when to replenish his stock and how much to buy. However, it is common in inventory management circumstances such as Mortimer's to reduce the question to two policy decisions: What **reorder point** level of inventory should trigger a decision to buy new stock, and what **order quantity** should be purchased each time? These two variables constitute our decisions. We presume that each time on-hand inventory falls below the reorder point, Mortimer will head to Antwerp to buy a standard reorder quantity.

The next issue is constraints. What restrictions limit MM's decision choices? In this example there aren't very many. It is only necessary that both decisions be nonnegative numbers and that the order quantity conform to the 100 carat minimum of the Antwerp market.

The third element is objectives. What makes one decision better than another? In MM's case the objective is clearly to minimize cost. More precisely, we want to minimize the sum of holding, replenishment, and lost-sales costs.

Summarizing in a verbal model or word description, our goal is *to choose a nonnegative reorder point and a nonnegative reorder quantity to minimize the sum of holding, replenishment, and lost-sales costs subject to the reorder quantity being at least* 100.

## *Optimization and Mathematical Programming*

Verbal models can help organize an analyst's thinking, but in this book we address a higher standard. We deal exclusively with optimization (also called mathematical programming).

> Definition 1.3 **Optimization models** (also called **mathematical programs**) represent problem choices as decision variables and seek values that maximize or minimize objective functions of the decision variables subject to constraints on variable values expressing the limits on possible decision choices.

With our Mortimer Middleman example, the decision variables are

$$q \triangleq \text{reorder quantity purchased on each replenishment trip}$$

$$r \triangleq \text{reorder point signaling the need for replenishment}$$

(Here and throughout $\triangleq$ means "is defined to be.") Constraints require only that

$$q \geq 100$$
$$r \geq 0$$

The objective function,

$$c(q, r) \triangleq \text{total cost using a reorder quantity of } q \text{ and a reorder point } r$$

remains to be explicitly represented mathematically. We seek to minimize $c(q, r)$ over values of $q$ and $r$ satisfying all constraints.

## *Constant-Rate Demand Assumption*

How we formulate constraints and objectives in terms of decision variables depends on what assumptions we are willing to make about the underlying system. We begin with a strong assumption regarding **constant-rate demand:** Assume that demand occurs at a constant rate of 55 carats per week. It is clear in Figure 1.1(a) that the demand rate is not exactly constant, but it does average 55 carats per week. Assuming that it is 55 carats in every week leads to some relatively simple analysis.

If the demand rate is constant, the pattern of on-hand inventory implied by a particular $q$ and $r$ will take one of the periodic "sawtooth" forms illustrated in Figure 1.3. Each time a shipment arrives, inventory will increase by order size $q$, then decline at the rate of 55 carats per week, producing regular cycles. Part (a) shows a case where inventory never runs out. A **safety stock** of (theoretically) untouched inventory protects against demand variability we have ignored. At the other extreme is part (c). Sales are lost because inventory runs out during the **lead time** between reaching the reorder point $r$ and arrival of a new supply. Part (b) has neither safety stock nor lost sales. Stock runs out just as new supply arrives.

## *Back of Envelope Analysis*

In cases where there are no lost sales [Figure 1.3(a) and (b)] it is easy to compute the length of each sawtooth cycle.

$$\frac{\text{order quantity}}{\text{demand rate}} = \frac{q}{55}$$

With lost sales [Figure 1.3(c)], each cycle is extended by a period when MM is out of stock that depends on both $q$ and $r$.

Clearly, both modeling and analysis would be easier if we could ignore the lost-sales case. Can we afford to neglect lost sales? As in so many OR problems, a bit of crude "back of envelope" examination of the relevant costs will help us decide.

Lost sales may occur under the best of plans because of week-to-week variation in demand. Under our constant-rate demand assumption, however,

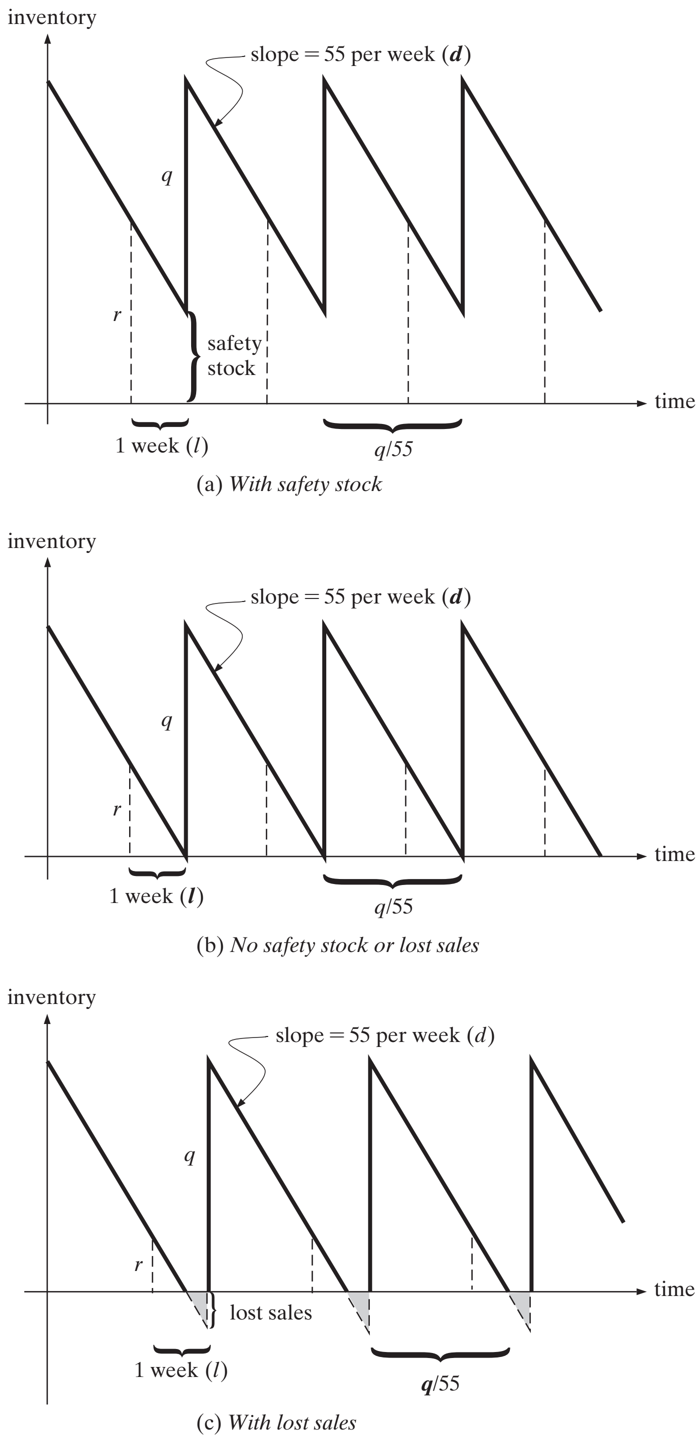



**FIGURE 1.3** Inventories Under Constant-Rate Demand

there is no variation. Furthermore, MM can afford to add a unit to $q$ and carry it for up to

$$\frac{\text{cost of lost sale}}{\text{weekly holding cost}} = \frac{\$200}{\$3.50} \approx 57.1 \text{ weeks}$$

rather than lose a carat of sales. Since the history in Figure 1.1 shows that inventory typically has been held no more than 4 to 6 weeks, it seems safe to make a second assumption regarding **no lost sales:** Assume that lost sales are not allowed.

## *Constant-Rate Demand Model*

Since customers order a constant-rate 55 carats during the 1 week it takes Mortimer to carry out an Antwerp trip, both inventory at order arrival and lost sales can be computed by comparing 55 to $r$. If $r < 55$, we lose $(55 - r)$ carats of sales each cycle, something we have decided not to permit. Thus we may deduce the constraint

$$r \geq 55$$

With $r$ restricted to be at least 55, $(r - 55)$ is the safety stock, and the cycle of rising and falling inventory repeats every $q/55$ weeks. Inventory on hand ranges from $(r - 55)$ at the low point of a cycle to $(r - 55) + q$ as a shipment arrives. The average will be the midpoint of these values, $(r - 55) + q/2$.

We are finally in a position to express all relevant costs. Holding cost per week is just the average inventory held times \$3.50. Replenishment cost per week is \$2000 divided by the cycle length or time between replenishments. Our first optimization model is

$$\begin{aligned} \text{minimize} \quad & c = 3.50\left[(r - 55) + \frac{q}{2}\right] + \frac{2000}{q/55} \\ \text{subject to} \quad & q \geq 100, \quad r \geq 55 \end{aligned} \tag{1.1}$$

## *Feasible and Optimal Solutions*

Remember that our goal is to help Mortimer make decisions. Since the decisions are the variables in our model, we would like to characterize good values for **decision variables** $q$ and $r$.

**Definition 1.4** A **feasible solution** is a choice of values for the decision variables that satisfies all constraints. **Optimal solutions** are feasible solutions that achieve objective function value(s) as good as those of any other feasible solutions.

For example, $q = 200, r = 90$ is feasible in constant-rate demand model (1.1) because both constraints are satisfied: $200 \geq 100$ and $90 \geq 55$.

Here we can go farther and find an optimal solution. To begin, notice that if $r$ deviates from demand 55, we incur extra holding cost and that no constraint prevents choosing $r$ exactly 55. We conclude that

$$r^* = 55$$

will tell MM the perfect moment to start travel preparations. The asterisk ($^*$) or **star** on a variable always denotes its optimal value.

Substituting this optimal choice of $r$ of (1.1), the objective function reduces to

$$c(q,r) \triangleq 3.50\left(\frac{q}{2}\right) + 2000\left(\frac{55}{q}\right) \tag{1.2}$$

Elementary calculus will tell us how to finish (differentiate with respect to $q$ and solve for a suitable point where the derivative is zero). To avoid being diverted by mathematical details in this introductory chapter, we leave the computation as an exercise for the reader.

The graphic presentation of cost function (1.2) in Figure 1.4 confirms the calculus result that the minimum average weekly cost occurs at

$$q^* = \pm\sqrt{\frac{2(2000)(55)}{3.50}} \approx 250.7$$

Since this value easily satisfies the $q \geq 100$ constraint, it is optimal.

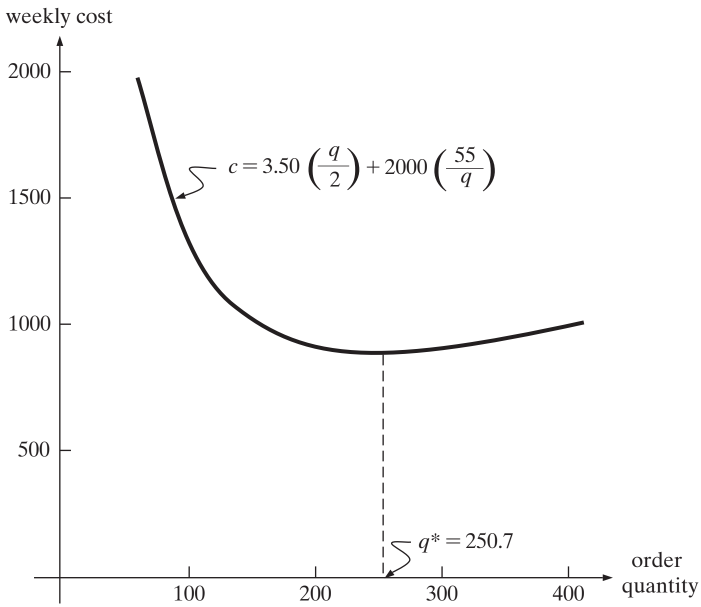



**FIGURE 1.4** Optimal MM Order Quantity Under Constant-Rate Demand

To summarize, our assumptions of constant-rate demand and no lost sales have led us to advise Mortimer to go to Antwerp whenever inventory drops below $r^* = 55$ carats and to buy $q^* = 250.7$ carats of new diamonds each trip. Substituting these values in the objective function of (1.1), total cost should be about \$877.50 per week or \$45,630 per year—quite an improvement over Mortimer's real experience of \$94,009.

## 1.3 System Boundaries, Sensitivity Analysis, Tractability, and Validity

The modeling in Section 1.2 took as given many quantities, such as the demand per week and the cost per carat held, then computed optimal values for reorder point and reorder quantity. A line between those items taken as settled and those to be decided is called the **system boundary**. Figure 1.5 illustrates how **parameters**—quantities taken as given—define objective functions and constraints applicable to the decision model inside. Together, parameters and decision variables determine results measured as **output variables**.

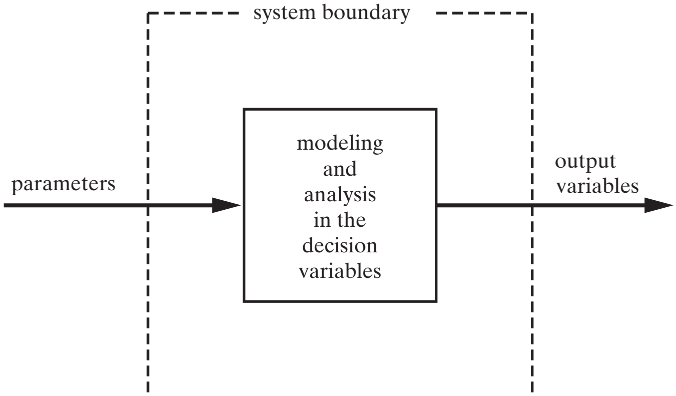



**Figure 1.5** System Boundaries

### *EOQ Under Constant-Rate Demand*

Only cost $c$ is an output variable in our constant-rate demand model of Mortimer Middleman's problem. Enumerating the parameters, let

$d \triangleq$ weekly demand (55 carats)

$f \triangleq$ fixed cost of replenishment ($2000)

$h \triangleq$ cost per carat per week for holding inventory ($3.50)

$s \triangleq$ cost per carat of lost sales ($200)

$\ell \triangleq$ lead time between reaching the reorder point and receiving a new supply (1 week)

$m \triangleq$ minimum order size (100 carats)

A great attraction of our constant-rate demand analysis is that it can be done just as well in terms of these symbols. If lost sales are not allowed, repetition of the analysis (calculus) in terms of symbolic parameters will cause us to conclude that

**Principle 1.5**

$$\text{optimal reorder quantity } q^* = \sqrt{\frac{2fd}{h}}$$
$$\text{optimal reorder point } r^* = \ell d$$